Solutions Tentamen Quantumfysica 1

Problem 1

a) A Hermitian ($A = A^+$) means that $\langle\phi|A|\psi\rangle = \langle A\phi|\psi\rangle$ for all ϕ, ψ. Equivalently, $\langle\phi|A|\phi\rangle$ should be real for all ϕ.

b) With $\psi = \psi(x,t)$,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi.$$

A stationary state is a state $\psi(x,t)$ which can be separated as

$$\psi(x,t) = T(t)u(x) = e^{-\frac{iEt}{\hbar}}u_E(x)$$

for some energy E. Equivalently, $\langle\psi|f(x,p)|\psi\rangle$ should be constant for all functions f of x and p.

c) Probability interpretation of quantum mechanics: $|\psi(x,t)|^2$ is a probability density. The total probability should be finite (after normalisation 1), so

$$\int |\psi(x,t)|^2 dx < \infty \ : \ \text{square integrable.}$$

d) Schrödinger picture: time dependence in wave functions. Heisenberg picture: time dependence in operators:

$$A(t) = e^{\frac{iHt}{\hbar}}A(0)e^{-\frac{iHt}{\hbar}}.$$

Problem 2

a) For $0 < x < L$ the Schrödinger equation solves as $u_1(x) = A\sin kx$, with $k^2 = \frac{2m}{\hbar^2}(E + V_0)$. For $x > L$, as $u_2(x) = Be^{-qx}$, with $q^2 = -\frac{2mE}{\hbar^2}$, so $k^2 = \frac{2mV_0}{\hbar^2} - q^2$. Matching u_1 and u_2 and their derivatives at $x = L$ gives $A\sin kL = Be^{-qL}$ and $kA\cos kL = -qBe^{-qL}$, so that $q\tan kL = -k$.

b) Minimum if $q \to 0$ but $k \not\to 0$, implying $k \to \frac{\pi}{2}$ (otherwise lhs vanishes), so $V_0 = \frac{\pi^2\hbar^2}{8ma^2}$.

Problem 3

a) $1 = \langle\psi(x,0)|\psi(x,0)\rangle = N^2(\langle u_0 + \cdots + u_4|u_0 + \cdots u_4\rangle) = N^2(\langle u_0|u_0\rangle + \cdots\langle u_4|u_4\rangle) = 5N^2$, so $N = \frac{1}{\sqrt{5}}$.

b) $\psi(x,t) = e^{-\frac{iHt}{\hbar}}\psi(x,0) = \frac{1}{\sqrt{5}}\left(e^{-\frac{iE_0t}{\hbar}}u_0(x) + \cdots + e^{-\frac{iE_4t}{\hbar}}u_4(x)\right).$

c) $\langle\psi(x,0)|P|\psi(x,0)\rangle = \frac{1}{5}\langle u_0 + u_1 + u_2 + u_3 + u_4|u_0 - u_1 + u_2 - u_3 + u_4\rangle = \frac{1}{5}(1 - 1 + 1 - 1 + 1) = \frac{1}{5}$.

d) $P(+1) = \sum |c(\text{states with } +1)|^2 = \frac{1}{5}(| \langle u_0|\psi(x,t)\rangle |^2 + | \langle u_2|\psi(x,t)\rangle |^2 + | \langle u_4|\psi(x,t)\rangle |^2) = \frac{3}{5}$.

e) $\psi_+(x,t) = \frac{1}{\sqrt{3}}(e^{-\frac{iE_0t}{\hbar}}u_0(x) + e^{-\frac{iE_2t}{\hbar}}u_2(x) + e^{-\frac{iE_4t}{\hbar}}u_4(x))$.

f) $P(E_0) = |c(E_0)|^2 = \frac{1}{3}| \langle u_0|\psi_+(x,t)\rangle |^2 = \frac{1}{3}$.

g) $Hu_n = (n+\frac{1}{2})\hbar\omega u_n$, so $\langle\psi(x,0)|H|\psi(x,0)\rangle = \frac{1}{5}\frac{\hbar\omega}{2}\langle u_0 + \cdots + u_4|u_0 + 3u_1 + 5u_2 + 7u_3 + 9u_4\rangle = \frac{5}{2}\hbar\omega$.

h) $\langle\psi_+|H|\psi_+\rangle = \frac{1}{3}\frac{\hbar\omega}{2}\langle u_0 + u_2 + u_4|u_0 + 5u_2 + 9u_4\rangle = \frac{5}{2}\hbar\omega$.

i) $\psi(x,t) = \frac{1}{\sqrt{2}}(\psi(x,0) - \psi(-x,0))$.

Problem 4

a) $(XY)^+ = Y^+X^+$, so $(B^+B)^+ = B^+B$, so $A^+ = A$.

b) $[B^+, B] = B^+B - BB^+ = A - 3 - (A-1) = -2$.

c) $[A, B] = [B^+B + 3, B] = [B^+B, B] = [B^+, B]B = -2B$.

d) $AB\psi = BA\psi + [A,B]\psi = Ba\psi - 2B\psi = (a-2)B\psi$, so $B\psi$ is an eigenfunction of A with eigenvalue $a-2$.

Problem 5

a) Correspondence principle: classical relations between variables hold for the expectation values of the corresponding quantum operators.

b) $\langle\cdot\rangle_t$ denotes $\langle\psi(x,t)|\cdot|\psi(x,t)\rangle$. $H = \frac{p^2}{2m}$, so

$$i\hbar\frac{d\langle x\rangle_t}{dt} = \langle[x,H]\rangle_t = \langle[x(t),H]\rangle_0 = \frac{1}{2m}\left\langle[x,p^2]\right\rangle_0 = \frac{1}{2m}\langle p[x,p] + [x,p]p\rangle_0 = \frac{i\hbar}{m}\langle p\rangle_0,$$

whence the statement.

c) Now $\langle\cdot\rangle_t$ denotes $\langle\phi(p,t)|\cdot|\phi(p,t)\rangle$ (different basis). $\phi(p,t) = e^{-\frac{iHt}{\hbar}}\phi(p,0)$; $x = i\hbar\frac{\partial}{\partial p}$, so

$$\begin{aligned}
\langle x\rangle_t &= \int \phi^*(p,t)x\phi(p,t)dp \\
&= i\hbar\int \phi(p,t)^*\frac{\partial}{\partial p}\phi(p,t)dp \\
&= i\hbar\int e^{\frac{ip^2t}{2m\hbar}}\phi^*(p,0)\frac{\partial}{\partial p}\left(e^{-\frac{ip^2t}{2m\hbar}}\phi(p,0)\right)dp \\
&= i\hbar\left(\int \phi^*(p,0)\frac{\partial\phi(p,0)}{\partial p}dp - \phi^*(p,0)\frac{-ipt}{m\hbar}\phi(p,0)dp\right) \\
&= \langle x\rangle_0 + \frac{\langle p\rangle_0}{m}t.
\end{aligned}$$

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